Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 3, pp. 369–380, 1996

0021–8928/96 $24.00+0.00

PII: S0021-8928(96)00047-0

BIFURCATION AND STABILITY OF THE STEADY MOTIONS AND RELATIVE EQUILIBRIA OF A RIGID BODY IN A CENTRAL GRAVITATIONAL FIELD†

Ye. V. ABRAROVA and A. V. KARAPETYAN

Moscow

(*Received* 26 *January* 1995)

The existence, bifurcation and stability of the steady motions of a rigid body in a central gravitational field are studied. The body is modelled as a collection of point masses situated at the ends of three mutually perpendicular diameters of a massless sphere. With this model, one can use the exact expression for the gravitational potential (see also [1–5]). The study considers non-trivial steady motions of a body with a triaxial ellipsoid of inertia such that either two or all three principal axes of inertia are not axes of the orbital coordinate system. In addition, a restricted formulation of the problem of the relative equilibria of a body whose mass centre is moving in a circular Keplerian orbit is considered, and the stability and bifurcation of these equilibria are investigated.

Previous publications [1, 2, 5] have investigated trivial steady motions corresponding to orientation of the body such that the principal central axes of inertia coincide with the axes of the orbital system of coordinates. Attention was devoted to the special case of a body with a spherical ellipsoid of inertia [3], and the special case of two-dimensional motions of the body [4].

1. We will consider the problem of the translational–rotational motion of a rigid body with a triaxial ellipsoid of inertia in a central gravitational field. We will model the body as a collection of point masses $m_s/2$ situated at the opposite ends of three mutually perpendicular diameters d_* ($s = 1, 2, 3$) of a massless sphere of radius a. Without loss of generality, we assume that $m_1 > m_2 > m_3$.

Let $O\xi\eta\zeta$ be a fixed system of coordinates with origin at the attracting centre, and let $Cx_1x_2x_3$ be a system of coordinates attached to the body with origin at its mass centre and axes along the diameters d_1, d_2, d_3. The position of the mass centre of the body relative to the fixed system of coordinates will be defined by spherical coordinates r, θ, ψ, where $r > a$ is the length of the radius vector $\mathbf{OC}$, θ is the angle between the vector $\mathbf{OC}$ and the plane $O\zeta\xi$ and is the angle between the axis $O\zeta$ and the projection of the vector $\mathbf{OC}$ onto the plane $O\zeta\xi$. The orientation of the orbit of the body's mass centre and the orientation of the body will be defined by the projections β_s and γ_s of the unit vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in the direction of the $O\eta$ axis and the radius vector $\mathbf{OC}$, respectively, onto the principal central inertia axes Cx_s ($s = 1, 2, 3$) of the body. Obviously, $\sin\vartheta = (\boldsymbol{\beta}, \boldsymbol{\gamma}) = \Sigma_s\beta_s\gamma$.

The kinetic energy T and the potential energy V of the body are

$$2T = [m(\dot{r}^2 + r^2\dot{\psi}^2\cos^2\theta + r^2\dot{\theta}^2) + J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2]$$

$$2V = -fM\sum_{s=1}^{3} m_s[F_s(a) + F_s(-a)], \quad F_s(a) = (r^2 + a^2 + 2ar\gamma_s)^{-1/2}$$

where $m = m_1 + m_2 + m_3$ is the mass of the body, $J_i = (m_j + m_k)a^2$ is the moment of inertia of the body about the axis Cx_i ($i \neq j \neq k; i, j, k \in S_3, S_3 = \{1, 2, 3\}$), is the projection of the absolute angular velocity ω of the body onto the axis Cx_s ($s = 1, 2, 3$), f is the gravitational constant, and m is the mass of the attracting centre.

The system admits of two first integrals: $H = T + V =$ const (energy), and $K = \partial T/\partial\dot{\psi} = k =$ const (area). Setting $\boldsymbol{\omega} = \dot{\psi}\boldsymbol{\beta} + \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity of the body relative to a system of coordinates rotating uniformly about the $O\eta$ axis, one finds the effective potential of the body (we recall that $\sin\theta = \Sigma\beta_s\gamma_s$, cf. [5])

$$W_k = \min_{\dot{r},\dot{\psi},\dot{\theta},\boldsymbol{\Omega}} H\Big|_{K=k} = V + \frac{k^2}{2J}, \quad J = mr^2\left(1 - \left(\sum_{s=1}^{3}\gamma_s\beta_s\right)^2\right) + \sum_{s=1}^{3}J_s\beta_s^2$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 3, pp. 375–387, 1996.

By Routh's theorem, the critical points $(r_0, \boldsymbol{\gamma}_0, \boldsymbol{\beta}_0)$ of the effective potential $W_k(r, \boldsymbol{\beta}, \boldsymbol{\gamma})$ on the manifold $\boldsymbol{\gamma}^2 = 1$, $\boldsymbol{\beta}^2 = 1$ are the steady motions of the body

$$r = r_0, \quad \boldsymbol{\gamma} = \boldsymbol{\gamma}_0, \quad \boldsymbol{\beta} = \boldsymbol{\beta}_0, \quad \dot\psi = \omega_0$$
$$(\theta = \theta_0 = \arcsin(\boldsymbol{\gamma}_0, \boldsymbol{\beta}_0), \quad \omega_0 = k / J_0, \quad \boldsymbol{\Omega} = \mathbf{0}) \tag{1.1}$$

Under conditions (1.1), the mass centre of the body uniformly describes a circle of radius $r_0 \cos\theta_0$ in a plane parallel to the plane $O\zeta\xi$ at a distance $r_0 \mid \sin\theta_0 \mid$ from it; the orientation of the body remains constant during the motion.

To determine the critical points of W_k on the manifold $\{\boldsymbol{\beta}^2 = 1, \boldsymbol{\gamma}^2 = 1\}$, we consider the function

$$W = (fM)^{-1} W_k + p(\boldsymbol{\gamma}^2 - 1)/2 + q(\boldsymbol{\beta}^2 - 1)/2$$

(p and q are undetermined Lagrange multipliers) and write down the conditions for this function to be stationary ($\varkappa^2 = k^2/(fM)$)

$$\frac{\partial W}{\partial r} = \frac{1}{2}\sum_{s=1}^{3} m_s((r + \alpha\gamma_s)F_s^3(a) + (r - a\gamma_s)F_s^3(-a)) - \frac{\varkappa^2 mr}{J^2}\left(1 - \left(\sum_{s=1}^{3}\gamma_s\beta_s\right)^2\right) = 0 \tag{1.2}$$

$$\frac{\partial W}{\partial \gamma_s} = \frac{1}{2} m_s (F_s^3(a) - F_s^3(-a)) ra + p\gamma_s + \frac{\varkappa^2 mr^2}{J^2}\beta_s \sum_{\sigma=1}^{3}\gamma_\sigma\beta_\sigma = 0 \tag{1.3}$$

$$\frac{\partial W}{\partial \beta_s} = \frac{\varkappa^2}{J^2}\left(mr^2\gamma_s \sum_{\sigma=1}^{3}\gamma_\sigma\beta_\sigma - J_s\beta_s\right) + q\beta_s = 0 \quad (s = 1,2,3) \tag{1.4}$$

Solutions of system (1.2)–(1.4) of the form

$$\gamma_i = \pm 1, \quad \beta_j = \pm 1, \quad (i \neq j \neq k), \quad \gamma_j = \gamma_k = \beta_i = \beta_k = 0 \quad (i, j, k \in S_3) \tag{1.5}$$

corresponding to trivial orientations of the body, have been investigated, and sufficient conditions have been established for the corresponding steady motions to be stable [5]

$$C_1 > 0, \quad C_2 > 0, \quad C_3 > 0, \quad C_4 = C_{44}C_{55} - C_{45}^2 > 0$$

$$C_1 = \frac{mr}{(mr^2 + J_j)^2}\frac{dK_{ij}}{dr}, \quad C_2 = \left[m_i \frac{(3r^2 + a^2)}{(r^2 - a^2)^3} - \frac{3m_k r}{(r^2 + a^2)^{5/2}}\right] ra^2$$

$$C_3 = \frac{\varkappa^2(J_j - J_k)}{(mr^2 + J_j)^2}, \quad C_{44} = \frac{\varkappa^2(J_j - J_i + mr^2)}{(mr^2 + J_j)^2}, \quad C_{45} = \frac{\varkappa^2 mr^2}{(mr^2 + J_j)^2}$$

$$C_{55} = \frac{\varkappa^2 mr^2}{(mr^2 + J_j)^2} + m_i ra^2 \frac{(3r^2 + a^2)}{(r^2 - a^2)^3} - \frac{3m_j r^2 a^2}{(r^2 + a^2)^{5/2}}$$

$$\varkappa^2 = K_{ij}(r), \quad K_{ij} = \frac{(mr^2 + J_j)^2}{mr}\left[m_i \frac{r^2 + a^2}{(r^2 - a^2)^2} + \frac{(m_j + m_k)r}{(r^2 + a^2)^{3/2}}\right]$$

$$(r_{ij}^+(\varkappa^2) > r_{ij}^0 > r_{ij}^-(\varkappa^2) > a, \quad r_{ij}^0 : K_{ij}'(r_{ij}^0) = 0;$$
$$\varkappa^2 > (\varkappa_{ij}^0)^2, \quad (\varkappa_{ij}^0)^2 = K_{ij}(r_{ij}^0))$$

In particular, it has been shown [4] that the degree of instability of the trivial steady motions may vary not only at the branch points $r = r_{ij}^0 (C_1(r_{ij}^0) = 0,\ i, j \in S_3)$ of the solutions (1.5) as functions of r, but also at the points $r = r_{ik}^* (C_2(r_{ik}^*) = 0, i > k)$, $r = \bar r_{ij}^* \ (C_4(\bar r_{ij}^*) = 0, i > j)$. This means that system (1.2)–(1.4), besides the solutions (1.5), has solutions corresponding to non-trivial orientations.

2. According to bifurcation theory, the solutions of system (1.2)–(1.4) bifurcate at the points r^*_{ik} = $(i > k)$ and $\bar{r}^*_{ij}$ $(i > j)$ as functions of γ_i and γ_k and as functions of γ_i, γ_j, β_i, β_j, respectively [4]. Thus, system (1.2)–(1.4) has solutions corresponding to orientations of the form

$$\gamma_i = \cos\varphi,\ \gamma_k = \sin\varphi,\ \gamma_j = \beta_i = \beta_k = 0,\ \beta_j = \pm 1\ \ (j \neq i > k \neq j) \tag{2.1}$$

$$\gamma_i = \cos\varphi,\ \gamma_j = -\sin\varphi,\ \beta_i = \sin(\varphi + \theta),\ \beta_j = \cos(\varphi + \theta),\ \gamma_k = \beta_k = 0\ \ (k \neq i > j \neq k) \tag{2.2}$$

Under conditions (2.1), the plane of the orbit of the body's mass centre passes through the attracting centre ($\sin\theta = 0$), the axis Cx_j points along the normal to the plane of the orbit, and the axes Cx_i and Cx_k are rotated through the same angle φ relative to the radius vector of the mass centre and the tangent to the orbit. The angle $\varphi = \varphi_{ik}(\varkappa^2)$ and radius of the orbit $r = r_{ik}(\varkappa^2)$ are determined from the system

$$m_k m_i^{-1} = \Phi_{ik}(r,\varphi),\quad \varkappa^2 = \frac{(mr^2 + J_j)^2}{2mr}\Psi_{ikj}(r,\varphi)$$

$$\Phi_{ik} = \frac{F_i^3(-a) - F_i^3(a)}{F_k^3(-a) - F_k^3(a)}\cdot\frac{\gamma_k}{\gamma_i},\quad \Psi_{ikj} = m_i G_i + m_k G_k + m_j \frac{2r}{(r^2 + a^2)^{3/2}} \tag{2.3}$$

$$G_s = F_s^3(a)(r + a\gamma_s) + F_s^3(-a)(r - a\gamma_s)\ \ (s = i,k);\ \ \gamma_i = \cos\varphi,\ \ \gamma_k = \sin\varphi$$

which is obtained from (1.2)–(1.4) by substituting conditions (2.1) into the system.

Proceeding as in [4], one can show that

$$\varphi = \pi n \pm \varphi_{ik}(\varkappa^2),\ (n = 0,1),\ \ r = r_{ik}(\varkappa^2),\ \ a < r_{ik}(\varkappa^2) < r^*_{ik}$$
$$0 < \varphi_{ik}(\varkappa^2) < \varphi^*_{ik} < \pi/4$$

where φ^*_{ik} is the unique root in the interval $0 < \varphi < \pi/2$ of the equation $m_k m_i^{-1} = \Phi_{ik}(a, \varphi)$, and r^*_{ik} is the unique root (for $r > a$) of the equation $C_2(r) = 0$.

Under conditions (2.2), the distance from the plane of the orbit of the body's mass centre to the attracting centre is $r\,|\sin\theta| \neq 0$, the radius of the orbit is $r\cos\theta$, the axis Cx_k is directed along the tangent to the orbit, the axis Cx_i makes an angle φ with the radius vector of the mass centre, and the axis Cx_j makes an angle $\varphi + \theta$ with the normal to the plane of the orbit. The angle θ may be expressed explicitly in terms of φ and the distance r from the mass centre to the attracting centre

$$\theta = \theta_{ij} = \frac{1}{2}\operatorname{arctg}\frac{\nu_{ij}(r)\sin 2\varphi}{1 - \nu_{ij}(r)\cos 2\varphi};\quad \nu_{ij} = \frac{m_i - m_j}{m}\frac{a^2}{r^2} \tag{2.4}$$

The angle φ and distance r are determined by solving a system analogous to system (2.3), but the functions Φ_{ij} and Ψ_{ijk} are more complicated.

Proceeding as before, it can be shown that

$$\varphi = \pi n \pm \bar{\varphi}_{ij}(\varkappa^2),\ (n = 0,1),\ \ r = \bar{r}_{ij}(\varkappa^2),\ \ a < \bar{r}_{ij}(\varkappa^2) < \bar{r}^*_{ij}$$
$$0 < \bar{\varphi}_{ij}(\varkappa^2) < \bar{\varphi}^*_{ij} < \varphi^*_{ij} < \pi/4 \tag{2.5}$$

where $\bar{\varphi}^*_{ij}$ is the unique root in the range $0 < \phi < \pi/2$ of the equation $m_j m_i^{-1} = \Phi_{ij}(a, \varphi)$, and $\bar{r}^*_{ij}$ is the unique root (for $r > a$) of the equation $C_4(r) = 0$.

The relative position of the points r^*_{ik}, $\bar{r}^*_{ij}$ and $r^0_{ik(j)}$ has a considerable effect on the degree of instability of the steady motions corresponding to orientations (1.5), (2.1) and (2.2), and depends on the relationships between the masses m_1, m_2 and m_3. Henceforth, for simplicity, we shall confine ourselves to the case of similar masses

$$m_2 = m_1(1 - u),\ \ m_3 = m_2(1 - v);\ \ 0 < u \ll 1,\ \ 0 < v \ll 1 \tag{2.6}$$

Under conditions (2.6), the ellipsoid of inertia is almost a sphere—the characteristic shape of many celestial bodies in nature.

Under these conditions one has $\overset{0}{r}_{ij(k)} \ll \bar{r}_{ij}^{*}(r_{ik}^{*})$ and, in addition, one can then show that the steady motions corresponding to orientations (2.1) and (2.2) exist when $\varkappa^2 \in ((\varkappa_{ik}^2)_*; (\varkappa_{ik}^2)^*$ and $\varkappa^2 \in ((\bar{\varkappa}_{ij}^{2})_*; (\bar{\varkappa}_{ij}^{2})^*)$, respectively, where $(\varkappa_{ik}^2)^*$ and $(\varkappa_{ik}^2)^*$ are defined by the second equation of system (2.3) with $r = a$, $\varphi = \varphi_{ik}^*$ or $r = r_{ik}^*$, $\varphi = 0$. The definitions of $(\bar{\varkappa}_{ij}^2)_*$ and $(\bar{\varkappa}_{ij}^2)^*$ are analogous.

3. To investigate the stability of the steady motions corresponding to orientations (2.1), we calculate the second variation of W over the linear manifold $\delta\beta_j = 0$, $\delta\gamma_i\cos\varphi + \delta\gamma_k\sin\varphi = 0$

$$\delta^2 W = \Sigma_1 + \Sigma_2$$

$$2\Sigma_1 = C_{11}(\delta r)^2 + 2C_{12}(\delta r)(\delta\gamma_k) + C_{22}(\delta\gamma_k)^2$$

$$2\Sigma_2 = C_{33}(\delta\gamma_j)^2 + 2C_{34}(\delta\gamma_j)(\delta\beta_i) + 2C_{35}(\delta\gamma_j)(\delta\beta_k) + $$
$$+ C_{44}(\delta\beta_i)^2 + 2C_{45}(\delta\beta_i)(\delta\beta_k) + C_{55}(\delta\beta_k)^2$$

$$C_{11} = \frac{mr}{(mr^2 + J_j)^2}\frac{\partial}{\partial r}\left[-\frac{1}{2}\frac{(mr^2 + J_j)^2}{mr}\Psi_{ikj}(r,\varphi)\right]$$

$$C_{12} = -\frac{3}{2}m_k\frac{ra}{\delta_i^{(3)}}\left[\delta_i^{(3)}(r\sigma_k^{(5)} + a\gamma_k\delta_k^{(5)}) - \delta_k^{(3)}(r\sigma_i^{(5)} + a\gamma_i\delta_i^{(5)})\right]$$

$$C_{22} = -\frac{1}{2}m_k\frac{ra}{\delta_i^{(3)}\gamma_i}\left[\frac{\delta_i^{(3)}\delta_k^{(3)}}{\gamma_i\gamma_k} + 3ra\sigma_k^{(5)}\delta_i^{(3)}\gamma_i + \sigma_i^{(5)}\delta_k^{(3)}\gamma_k\right]$$

$$C_{33} = \frac{1}{2}r^2\left[m_i\sigma_i^{(3)} + m_k\sigma_k^{(3)} + \frac{2m_j(r^2 - 2a^2)}{(r^2 + a^2)^{5/2}}\right]$$

$$C_{44} = \frac{\varkappa^2}{J^2}\left[mr^2\cos^2\varphi + (J_j - J_i)\right], \quad C_{55} = \frac{\varkappa^2}{J^2}\left[mr^2\sin^2\varphi + (J_j - J_k)\right]$$

$$C_{34} = \pm\frac{\varkappa^2}{J^2}mr^2\cos\varphi, \quad C_{35} = \pm\frac{\varkappa^2}{J^2}mr^2\sin\varphi, \quad C_{45} = \pm\frac{\varkappa^2}{J^2}mr^2\cos\varphi\sin\varphi$$

$$\delta_s^{(n)} = F_s^n(a) - F_s^n(-a), \quad \sigma_s^{(n)} = F_s^n(a) + F_s^n(-a), \quad (s = 1,2,3;\ n = 3,5)$$

where $\gamma_i = \cos\varphi$, $\gamma_k = \sin\varphi$ and r and φ satisfy the first relations of system (2.3) and depend on $\varkappa^2$ (see the second eqŭation of that system); $i, j, k \in S_3$, $i > k$.

Since $C_{22} < 0$, all steady motions corresponding to orientations (2.1) are unstable in the secular sense. The degree of instability of these motions when $r < r_{ik}^*$, close to r_{ik}^* is identical with the degree of instability of the corresponding trivial steady motions when $r > r_{ik}^*$ ($i > k$), and does not vary along the entire branch (2.3) ($a < r_{ik}(\varkappa^2) < r_{ik}^*$) if the determinant Δ of the quadratic form $\delta^2 W = \Sigma_1 + \Sigma_2$ does not change sign for all $r \in (a; r_{ik}^*)$, $\varphi \in (0; \varphi_{ik}^*)$. Since $\Delta = \Delta_1\Delta_2$, where $\Delta_{1,2}$ is the determinant of the quadratic form $\Sigma_{1,2}$, and moreover $\Delta_1 \neq 0$ for all steady motions corresponding to orientations (2.1), it follows that Δ vanishes if and only if Δ_2 vanishes. The determinant Δ_2 does not vanish for orientations (2.1) when $i = 2, j = 3, k = 1$; it vanishes when $i = 3, j = 1, k = 2$, and may vanish when $i = 3, j = 2, k = 1$ at some point $(r_{3k}^{**}; \varphi_{3k}^{**})$, $a < r_{3k}^{**} < r_{3k}^*$, $0 < \varphi_{3k}^{**} < \varphi_{3k}^*$ ($k = 1, 2$). For $k = 1, i = 2, j = 3$, this point exists for arbitrary values of the masses $m_1 > m_2 > m_3$, but for $k = 2, j = 3, j = 1$ it exists only when $2m_2 > m_1 + m_3$ (i.e. only when $\nu > u$; see (2.6)). Under those conditions the degree of instability of the steady motions for orientations (2.1) ($k = 1, 2$) is one less for $a < r < r_{3k}^{**}$ than for $r_{3k}^{**} < r < r_{3k}^*$.

To investigate the stability of the steady motions corresponding to orientations (2.2), one must evaluate the second variation of W over the linear manifold

$$\delta\gamma_i\cos\varphi - \delta\gamma_j\sin\varphi = 0, \quad \delta\beta_j\cos(\theta + \varphi) + \delta\beta_i\sin(\theta + \varphi) = 0$$

$$\delta^2 W = \Sigma_1 + \Sigma_2$$

$$2\Sigma_1 = C_{11}(\delta r)^2 + 2C_{12}(\delta r)(\delta\gamma_j) + 2C_{13}(\delta r)(\delta\beta_i) + C_{22}(\delta\gamma_j^2) + 2C_{23}(\delta\gamma_j)(\delta\beta_i) + C_{33}(\delta\beta_i)^2$$

$$2\Sigma_2 = C_{44}(\delta\gamma_k)^2 + 2C_{45}(\delta\gamma_k)(\delta\beta_k) + C_{55}(\delta\beta_k)^2$$

(the coefficients C_{pq} are very cumbersome in form and are therefore not given here).

As before, it can be shown that all steady motions corresponding to orientations (2.2) are unstable in the secular sense, since $\Delta_1 < 0$. When $i = 2, j = 1, k = 3$, the degree of instability of these motions does not vary ($\Delta_1 \neq 0, \Delta_2 \neq 0$) along the entire branch (2.5) and is identical with the degree of instability of the trivial steady motions for $r > r_{21}^*$. When $i = 3, j = 2, k = 1$ the degree of instability varies, while when $i = 3, j = 2, k = 1$ it may vary at some point $(\bar{r}_{3j}^{**}; \bar{\varphi}_{3j}^{**})$, $a < \bar{r}_{3j}^{**} < \bar{r}_{3j}^*$, $0 < \bar{\varphi}_{3j}^{**} < \bar{\varphi}_{3j}^*$ $(\Delta_2(\bar{r}_{3j}^{**}; \bar{\varphi}_{3j}^{**}) = 0, j = 1, 2$. When $j = 2$ this point exists for arbitrary masses $m_1 > m_2 > m_3$, but when $j = 1$ it exists only when $2m_2 > m_1 + m_3$, i.e. when $v > u$ (as in the previous case $\Delta_1 \neq 0$ for all orientations (2.2)). Under these conditions the degree of instability of the steady motions for orientations (2.2) ($j = 1, 2$) when $\bar{r}_{3j}^{**} < r < \bar{r}_{3j}^*$ is identical with the degree of instability of the corresponding trivial steady motions when $r > \bar{r}_{3j}^*$, while when $a < r < \bar{r}_{3j}^{**}$ ($j = 1, 2$) it is one less than the latter.

At the points $(r_{32}^{**}; \varphi_{32}^{**})$ and $(\bar{r}_{32}^{**}; \bar{\varphi}_{32}^{**})$ there will always be steady motions bifurcating from the motions corresponding to orientations (2.1) ($i = 3, j = 1, k = 2$) and (2.2) ($i = 3, j = 2, k = 1$); these are motions corresponding to orientations of the general form

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^0, \quad \boldsymbol{\beta} = \boldsymbol{\beta}^0 \quad ((\boldsymbol{\gamma}^0, \boldsymbol{\beta}^0) \neq 0, \quad \gamma_s^0 \neq 0, \quad \beta_s^0 \neq 0, \quad \beta_s^0 \neq 0, \quad \forall s = 1, 2, 3) \tag{3.1}$$

If in addition the condition $2m_2 > m_1 + m_3$ is satisfied, the analogous assertion is true when ($i = 3$, $j = 2, k = 1$), ($i = 3, j = 1, k = 2$) also, for orientations (2.1) and (2.2), respectively. It can be shown that these steady motions exist only when $a < r < \bar{r}_{3k}^{**}$, $a < r < \bar{r}_{3j}^{**}$ and only when $\varkappa^2 < (\varkappa_{3k}^2)^{**}$, $\varkappa^2 < (\varkappa_{3j}^2)^{**}$, respectively, where

$$(\varkappa_{3k}^2)^{**} = \left[\frac{(mr^2 + J_j)^2}{2mr}\Psi_{3kj}(r, \varphi)\right]_{r = r_{3k}^{**}, \varphi = \varphi_{3k}^{**}} \quad (k, j = 1, 2; k \neq j)$$

$$(\varkappa_{3j}^2)^{**} = \left[\frac{(mr^2 + J_j)^2}{2mr}\bar{\Psi}_{3jk}(r, \varphi)\right]_{r = \bar{r}_{3j}^{**}, \varphi = \bar{\varphi}_{3j}^{**}}$$

Steady motions of general form are determined from system (1.2)–(1.4). They are characterized by the fact that the plane of the orbit of the body's mass centre does not pass through the attracting centre (as in case (2.2)), and moreover none of the principal central axes of inertia of the body coincides with any of the axes of the orbital system of coordinates (unlike cases (1.5), (2.1) and (2.2)).

4. Equations (1.2)–(1.4) define a single-parameter family of steady motions of the body (the curve $L = \{r = (\varkappa^2), \gamma = \gamma(\varkappa^2), \beta = \beta(\varkappa^2)\}$ in the space $(r, \gamma, \beta, \varkappa^2)$). Sections of this space by the hyperplanes (1.5) are shown in Figs 1–6. The solid curves correspond to those branches of L lying in the hyperplanes and corresponding to trivial steady motions. The dashed and dash–dot curves correspond to the projections of those branches of L that leave the hyperplanes and correspond to steady motions (2.1) and (2.2). The dotted curves correspond to the projections of those branches of L that always leave the aforementioned non-trivial branches and correspond to orientations (3.1). Figure 5 corresponds to the case when $3v > u > v$ (see (2.6)); when $3v < u$ one must interchange the dotted and dash–dot curves in Fig. 5. When $v > u$ one must add to Figs 4 and 5 curves leaving the non-trivial branches and corresponding to additional steady motions of type (3.1). The digits 0, 1, 2 and 3 indicate the degree of instability of the steady motions of the body corresponding to the relevant orientations (1.5), (2.1), (2.2) or (3.1). The degree of instability of the latter is indicated in accordance with the general considerations of bifurcation theory.

5. We will now consider a restricted formulation of the problem, assuming that, independently of the rotational motions of the body, its mass centre moves uniformly along a circular orbit of radius $r_0 \gg a$ ($r_0 = \text{const}$) situated in the $O\xi\zeta$ plane. Then $\theta = 0$, and, generally speaking, the system admits of only a generalized energy integral $\dot{\psi} \equiv \omega_0 = (fMm/r_0^3) = \text{const}$, where $T_2^0 - T_0^0 + V = \text{const}$ are second-degree and zero-degree forms in the velocities, the constituents of $T^0 = T_2^0 + T_1^0 + T_0^0$ ($T^0 = T|_{r_0 = r, \theta = 0, \psi = \omega_0}$). As in Section 1, setting $\boldsymbol{\omega} = \omega_0\boldsymbol{\beta} + \boldsymbol{\Omega}$, we can write the changed potential of the body as

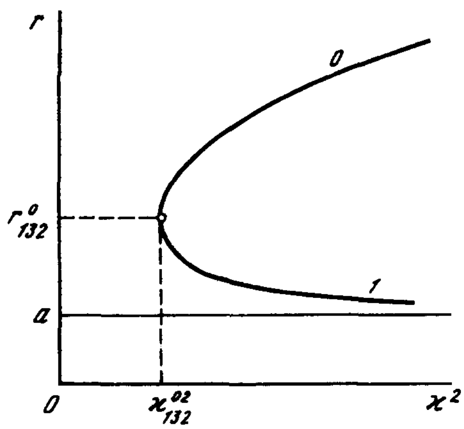

Fig. 1.

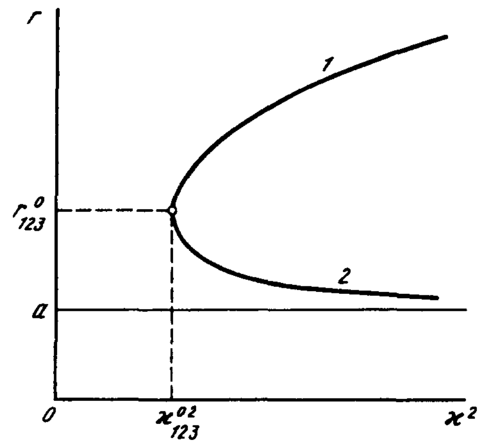

Fig. 2.

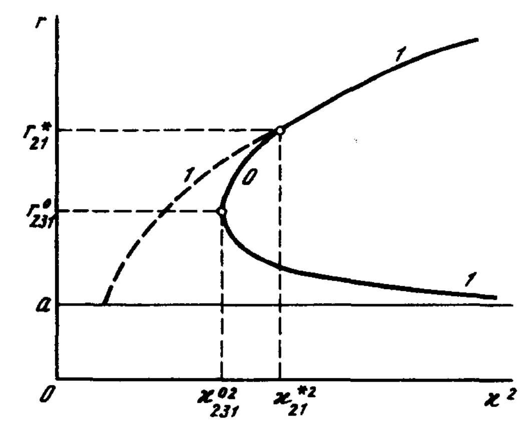

Fig. 3.

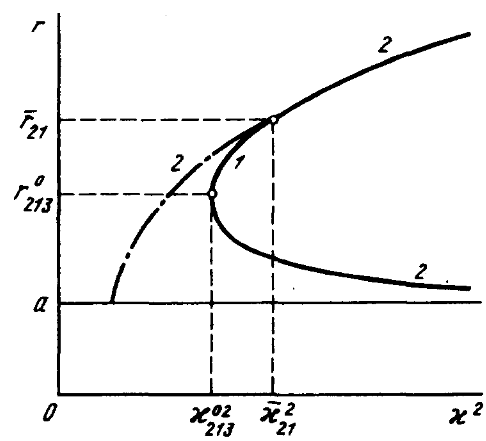

Fig. 4.

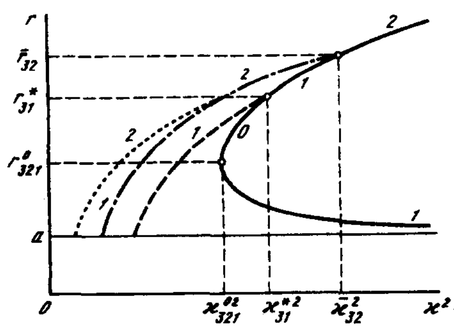

Fig. 5.

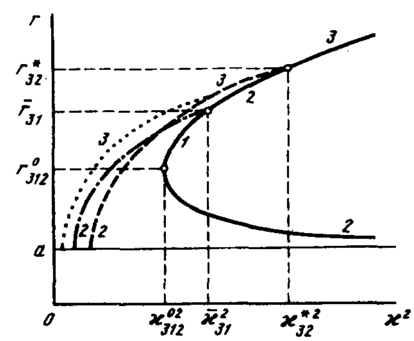

Fig. 6.

$$W_\omega^0 = V - J\omega_0^2/2$$

$$W_\omega^0 = -\frac{fM}{2}\sum_{s=1}^{3} m_s(F_s(a)+F_s(-a)) - \frac{1}{2}\omega_0^2(J_1\beta_1^2 + J_2\beta_2^2 + J_3\beta_3^2)$$

To the critical points (γ_0, β_0) of the varied potential W_ω^0 on the manifold $\{\boldsymbol{\gamma}^2 = 1;\ \boldsymbol{\beta}^2 = 1;\ (\boldsymbol{\gamma}\cdot\boldsymbol{\beta}) = 0\}$ there correspond relative equilibria of the body in its circular orbit. To search for these points, we define a function

$$W^0 = (fM)^{-1}W_\omega^0 + \lambda(\boldsymbol{\gamma}\cdot\boldsymbol{\beta}) + \sigma(\boldsymbol{\gamma}^2 - 1)/2 + \nu(\boldsymbol{\beta}^2 - 1)/2$$

(λ, σ and ν are undetermined Lagrange multipliers) and write down the conditions for it to be stationary (assuming, without loss of generality, that $fM = 1$, $m = 1$, $r_0 = 1$; then $\omega_0 = 1$ and $a \ll 1$)

$$\partial W^0/\partial\gamma_s = (am_s)(F_s^3(a) - F_s^3(-a))/2 + \lambda\beta_s + \sigma\gamma_s = 0 \tag{5.1}$$

$$\partial W^0/\partial\beta_s = \beta_s(\nu - J_s) + \lambda\gamma_s = 0 \quad (s = 1,2,3)$$

System (5.1) admits of solutions

$$\gamma_i = \pm 1;\quad \beta_j = \pm 1;\quad \gamma_j = \gamma_k = \beta_i = \beta_k = 0 \quad (i \neq j \neq k) \tag{5.2}$$

which correspond to trivial relative equilibria of the body (in which case $\lambda = 0$) analogous to the steady motions (1.5).

To investigate the stability of these relative equilibria, we will calculate the second variation of W^0 over the linear manifold

$$\delta\gamma_i = \delta\beta_j = 0;\quad \delta\gamma_j = -\delta\beta_i$$

We have

$$2\delta^2 W = C_1^0(\delta\gamma_j)^2 + C_2^0(\delta\gamma_k)^2 + C_3^0(\delta\beta_k)^2$$

$$C_1^0 = a^2(m_i - m_j + bm_i - cm_j),\quad C_2^0 = a^2(m_i b - m_k c)$$

$$C_3^0 = a^2(m_k - m_j);\quad b = \frac{(3+a^2)}{(1-a^2)^3},\quad c = \frac{3}{(1+a^2)^{5/2}}$$

Obviously, the conditions for stability of the relative equilibria (5.2) are $C_1^0 > 0$, $C_2^0 > 0$, $C_3^0 > 0$.

If $m_i > m_j$, then $C_1^0 > 0$; but if $m_i < m_j$, then $C_1^0 > 0$ ($C_1^0 < 0$) for $\mu_2 \equiv m_j/m_i < \mu_{ij}$ ($\mu_2 > \mu_{ji}$). If $m_i > m_k$, then $C_2^0 > 0$; but if $m_i < m_k$, then $C_2^0 > 0$ ($C_2^0 < 0$) for $\mu_1 \equiv m_k/m_i < \mu_{ki}$ ($\mu_1 > \mu_{ki}$). Finally, $C_3^0 > 0$ ($C_3^0 < 0$) for $m_k > m_j$ ($m_k < m_j$). We have used the following notation

$$\mu_{ji} \equiv \mu_{ji}(a) = \frac{(4 - 2a^2 + 3a^4 + a^6)}{(c+1)(1-a^2)^3} = 1 + \frac{35}{8}a^2 + o(a^2)$$

$$\mu_{ki} \equiv \mu_{ki}(a) = \frac{b}{c} = 1 + \frac{35}{6}a^2 + o(a^2)$$

Thus, the relative equilibria (5.2) are

(a) always unstable if $i = 1, j = 2, k = 3$ ($m_i > m_j > m_k$; $\mu_1 < \mu_2 < 1$) (the degree of instability $\chi = 1$);

(b) always stable if $i = 1, j = 3, k = 2$ ($m_i > m_k > m_j$; $\mu_2 < \mu_1 < 1$) (then $\chi = 0$);

(c) when $i = 2, j = 1, k = 3$ ($m_j > m_i > m_k$; $\mu_2 > 1 > \mu_1$), they are unstable ($\chi = 1$) if $\mu_2 < \mu_{ji}$, and unstable in the secular sense ($\chi = 2$) if $\mu_2 > \mu_j$;

(d) when $i = 2, j = 3, k = 1$ ($m_k > m_i > m_j$; $\mu_1 > 1 > \mu_2$), they are stable ($\chi = 0$) if $\mu_1 < \mu_{ki}$ and unstable ($\chi = 1$) if $\mu_1 < \mu_{ki}$;

(e) when $i = 3, j = 1, k = 2$ ($m_j > m_k > m_i$; $\mu_2 > \mu_1 > 1$), they are unstable if $\mu_2 < \mu_{ji}$ ($\chi = 1$) or $\mu_1 > \mu_{ki}$ ($\chi = 3$), and unstable in the secular sense ($\chi = 2$) if $\mu_1 < \mu_{ki}$ and $\mu_2 > \mu_{ji}$;

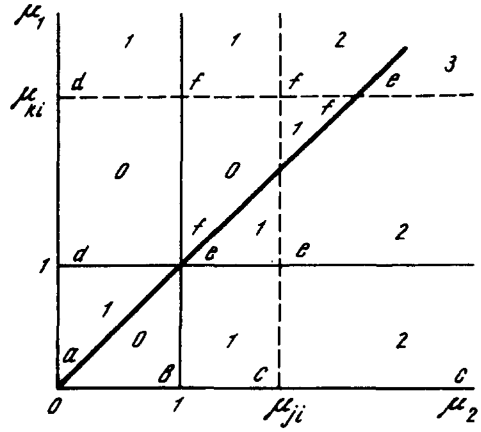

Fig. 7.

(f) when $i = 3, j = 2, k = 1$ ($m_k > m_j > m_i$; $\mu_1 > \mu_2 > 1$), they are stable ($\chi = 0$) if $\mu_1 < \mu_{ki}$ and $\mu_2 < \mu_{ji}$, unstable ($\chi = 1$) if $\mu_1 > \mu_{ki}$ and $\mu_2 < \mu_{ji}$, or $\mu_1 < \mu_{ki}$ and $\mu_2 > \mu_{ji}$, and unstable in the secular sense ($\chi = 2$) if $\mu_1 > \mu_{ki}$ and $\mu_2 > \mu_{ji}$.

Figure 7 shows the plane of the parameters μ_1 and μ_2 divided by solid lines into six domains corresponding to cases (a)–(f), respectively, with digits 0–3 indicating the degree of instability of the corresponding relative equilibria (5.2). The dashed lines indicate the bifurcation lines $\mu_1 = \mu_{ki}$ and $\mu_2 = \mu_{ji}$ across which the degree of instability of the trivial orientations changes (in cases (c)–(f)).

6. The degree of instability of the trivial relative equilibria (5.2) corresponding to cases (d)–(f) changes when $\mu_1 = \mu_{ki}$. When that happens the coefficient C_2^0 vanishes and the following solutions bifurcate from the aforementioned trivial solutions of system (5.1) ($i > k$)

$$\gamma_i = \cos\varphi_0;\quad \gamma_k = \sin\varphi_0;\quad \beta_j = \pm 1;$$
$$\gamma_j = \beta_i = \beta_k = 0 \tag{6.1}$$

The degree of instability of the trivial relative equilibria corresponding to cases (c), (e) and (f) changes when $\mu_2 = \mu_{ji}$. When that happens the coefficient C_1^0 vanishes and the following solutions bifurcate from the aforementioned trivial solutions of system (5.1)

$$\gamma_i = \cos\psi_0;\quad \gamma_j = -\sin\psi_0;\quad \beta_i = \sin\psi_0;\quad \beta_j = \cos\psi_0;\quad \gamma_k = \beta_k = 0 \tag{6.2}$$

Note that for the solutions (6.1) $\mu = 0$, and for the solutions (6.2) $\lambda = a^2 \sin\psi \cos\psi\,(m_j - m_i) \neq 0$. The relative equilibria (6.1) are analogous to the steady motions (2.1) for which $\theta = 0$, but the relative equilibria (6.2) are essentially distinct from the corresponding steady motions (2.2), for which $\theta \neq 0$. Moreover, since $\lambda \neq 0$ for the solutions (6.2), a necessary condition for their existence is the application of forces that keep the mass centre of the body in the plane containing the attracting centre.

The angles φ_0 and ψ_0 are found from the equations

$$\mu_1 = \operatorname{tg}\varphi_0 \frac{F_i^3(-a) - F_i^3(a)}{F_k^3(-a) - F_k^3(a)} \tag{6.3}$$

$$\mu_2 = \operatorname{tg}\psi_0 \frac{F_i^3(-a) - F_i^3(a) + 2a\cos\psi_0}{F_j^3(a) - F_j^3(-a) + 2a\sin\psi_0} \tag{6.4}$$

(compare (6.3) with the first equation of system (2.3)). The properties of the solutions of Eqs (6.3) and (6.4) are analogous to those of the solutions (2.1) and (2.2).

Solutions (6.1) and (6.2) exist only when $\mu_1 < \mu_{ki}$ and $\mu_2 < \mu_{ji}$, respectively (the ellipsoid of inertia of the body is almost an ellipsoid of revolution); henceforth, therefore, we shall assume, without loss of generality, that

$$m_k = m_i(1+ha^2)+o(a^2), \quad m_j = m_i(1+ga^2)+o(a^2) \tag{6.5}$$

7. To investigate the stability of the relative equilibria (6.1), we will calculate the second variation of W^0 over the linear manifold

$$\delta\beta_j = 0; \quad \delta\gamma_i = -\operatorname{tg}\varphi_0\delta\gamma_k; \quad \delta\gamma_j = -\cos\varphi_0\delta\beta_i - \sin\varphi_0\delta\beta_k$$

We have

$$2\delta^2 W^0 = C_{11}^0(\delta\gamma_k)^2 + C_{22}^0(\delta\beta_i)^2 + 2C_{23}^0\delta\beta_i\delta\beta_k + C_{33}^0(\delta\beta_k)^2$$

$$C_{11}^0 = a(m_i[F_i^3(-a) - F_i^3(a)]/\cos\varphi_0 - 3a[\sin^2\varphi_0 m_i[F_i^5(a) + F_i^5(-a)] -$$
$$-\cos^2\varphi_0 m_k[F_k^5(a) + F_k^5(-a)]])/2\cos^2\varphi_0$$

$$C_{22}^0 = a^2(m_i - m_j) + a\cos^2\varphi_0\delta_{ij}, \quad C_{23}^0 = a\sin\varphi_0\cos\varphi_0\delta_{ij}$$

$$C_{33}^0 = a^2(m_k - m_j) + a\sin^2\varphi_0\delta_{ij}$$

$$\delta_{ij} = m_i[F_i^3(-a) - F_i^3(a)]/2\cos\varphi_0 - 3cm_j$$

Taking the first relation in (6.5) into account, we conclude that $C_{11}^0 < 0$ for all $i > k$, i.e. all the relative equilibria (6.1) are unstable in the secular sense. A detailed analysis of the other coefficients of the quadratic form $\delta^2 W_0$ shows that the relative equilibrium is:

(d) always unstable when $i = 2, j = 3, k = 1$ (then $\chi = 1$);

(e) when $i = 3, j = 1, k = 2$, it is unstable ($\chi = 3$) if $g > g_+$ and unstable in the secular sense ($\chi = 2$) if $g < g_+$;

(f) when $i = 3, j = 2, k = 1$, it is unstable ($\chi = 1$) if $g < g_-$ and unstable in the secular sense ($\chi = 2$) if $g > g_-$.

We have used the notation

$$g_\pm = g_\pm(\varphi_0) = \frac{35}{48}\left[6\cos^4\varphi_0 + \cos 2\varphi_0 \pm \left(\frac{9}{4}\sin^4 2\varphi_0 + \cos^4 2\varphi_0\right)^{1/2}\right]$$

Similarly, one can investigate the stability of the relative equilibria (6.2) by analysing the second variation of W^0 over the linear manifold $\delta\gamma_j = -\delta\beta_i$, $\delta\gamma_i = \delta\beta_j = -\operatorname{tg}\psi_0\delta\beta$.

It turns out that all the relative equilibria (6.2) ($i > j$) are unstable in the secular sense, and the equilibrium (6.2) is:

(c) unstable in the secular sense when $i = 2, j = 1, k = 3$ (then $\chi = 2$);

(e) when $i = 3, j = 1, k = 2$, it is unstable ($\chi = 1$) if $h > h_-$ and unstable in the secular sense ($\chi = 2$) if $h < h_-$;

(f) when $i = 3, j = 2, k = 1$, it is unstable ($\chi = 1$) if $h < h_+$ and unstable in the secular sense ($\chi = 2$) if $h > h_+$.

We have used the notation

$$h_\pm = h_\pm(\psi_0) = \frac{35}{48}[8\cos^4\psi_0 - \cos 2\psi_0 \pm (4\sin^4 2\psi_0 + \cos^4 2\psi_0)^{1/2}]$$

The solid lines in Fig. 8 define the domain in the μ_1, μ_2 plane in which the solutions (6.1) and (6.2) exist. The solutions (6.1) exist in the strip $\mu_1 \in (1; \mu_{ki})$, $\mu_2 > 0$, while the solutions (6.2) exist in the strip $\mu_2 \in (1; \mu_{ji})$, $\mu_1 > 0$. The first strip is divided by the small circles into three domains, corresponding to cases (6.1) (d)–(f) (marked d_1–f_1), while the second strip is divided by crosses into the domains corresponding to cases (6.2) (c), (e) and (f) (marked c_2, e_2, f_2).

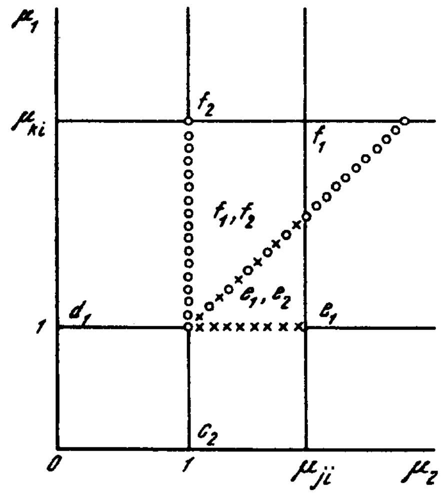
μ_1
μ_{ki}
f_2
f_1
f_1, f_2
e_1, e_2
d_1
e_1
1
c_2
0
1
μ_{ji}
μ_2

Fig. 8.

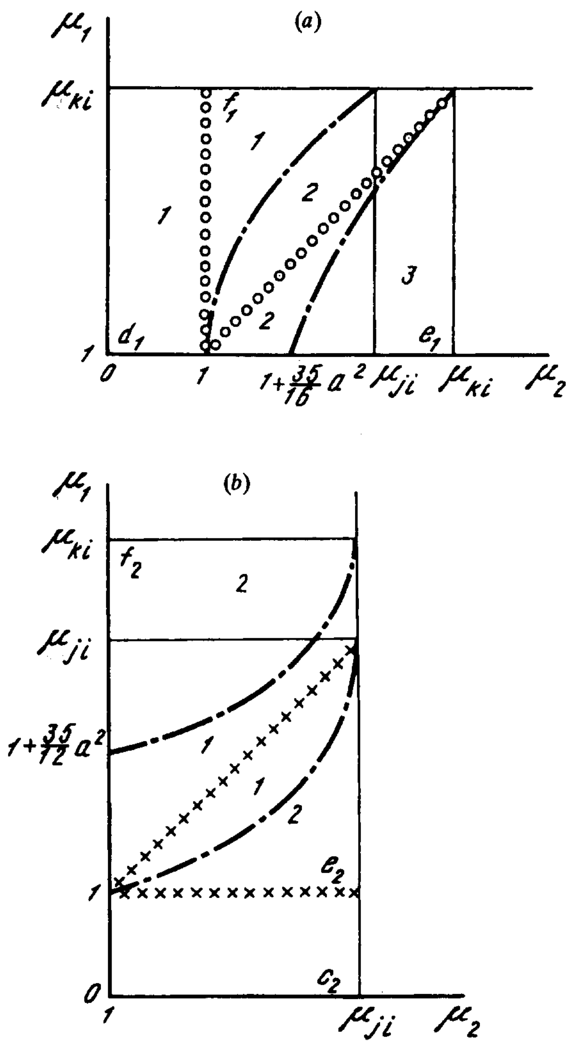
(a)
μ_1
μ_{ki}
f_1
1
1
2
3
2
d_1
e_1
1
0
1
$1+\frac{35}{16}a^2$
μ_{ji}
μ_{ki}
μ_2
(b)
μ_1
μ_{ki}
f_2
2
μ_{ji}
$1+\frac{35}{12}a^2$
1
1
2
e_2
1
0
c_2
1
μ_{ji}
μ_2

Fig. 9.

These strips are shown separately in Fig. 9, with the degrees of instability of the respective relative equilibria (6.1) and (6.2) indicated. The dash–dot curves correspond to projections of the bifurcation curves across which the degree of instability of the equilibrium orientations (6.1) and (6.2) changes (in cases (e) and (f)).

8. In cases (e) and (f) ($i = 3, j, k = 1, 2$) there is a change in the degree of instability of the relative equilibria (6.1) (when $g = g_\pm$) and (6.2) (when $h = h_\pm$). In these cases solutions of the general form

$$\begin{aligned}&\gamma_i = \cos\varphi_0\cos\psi_0;\quad \gamma_j = \sin\varphi_0\cos\psi_0\sin\theta_0 - \sin\psi_0\cos\theta_0\\&\gamma_k = \sin\varphi_0\cos\psi_0\cos\theta_0 + \sin\psi_0\sin\theta_0\\&\beta_i = \cos\varphi_0\sin\psi_0;\quad \beta_j = \sin\varphi_0\sin\psi_0\sin\theta_0 + \cos\psi_0\cos\theta_0\\&\beta_k = \sin\varphi_0\sin\psi_0\cos\theta_0 - \sin\theta_0\cos\psi_0\\&\left(\lambda = (J_i - J_j)\sin\psi_0\left(\cos\psi_0 + \frac{\sin\varphi_0\sin\theta_0\sin\psi_0}{\cos\theta_0}\right) \neq 0\right)\end{aligned} \tag{8.1}$$

bifurcate.

These solutions exist only for $i = 3$ and only when the conditions $\mu_1 < \mu_{k3}$, $\mu_2 < \mu_{k3}$ hold simultaneously; these conditions are equivalent to conditions (6.5) and mean that the ellipsoid of inertia of the body is almost a sphere. These relative equilibria differ essentially from the corresponding steady motions (3.1), for which $\theta \neq 0$; they are always unstable in the secular sense (depending on the relationships between the masses m_1, m_2 and m_3, in accordance with (6.5), the degree of instability of the solutions (8.1) may equal 1, 2 or 3).

The solid curves in Fig. 10 indicate the domains of existence of the solutions (8.1); the digits 1–3 indicate their degrees of instability (depending on the positions of the bifurcation points).

9. Hence, the investigation of our model problem has revealed the following phenomenon, due to the use of an exact expression for the potential of the gravitational forces

1. the existence of secular stability of the steady motions and relative equilibria of the body corresponding to trivial orientations, in cases in which not only the major axis of the body's ellipsoid of inertia ($i = 1, j = 3, k = 2$) but also the median ($i = 2, j = 3, k = 1$) and minor ($i = 3, j = 2, k = 1$) axes point along the radius vector of its mass centre (see also [1, 4, 5]);
2. the existence of non-trivial steady motions and relative equilibria for which at least two of the principal central axes of inertia are no axes of the orbital system of coordinates (see also [4]);
3. the existence of steady motions of the body for which the plane of the orbit of the mass centre does not pass through the attracting centre.

It should be noted that if the ellipsoid of inertia is almost a sphere (see (2.6)), the bifurcation values of the radius of the orbit of the mass centre in the trivial steady motions are defined as follows:

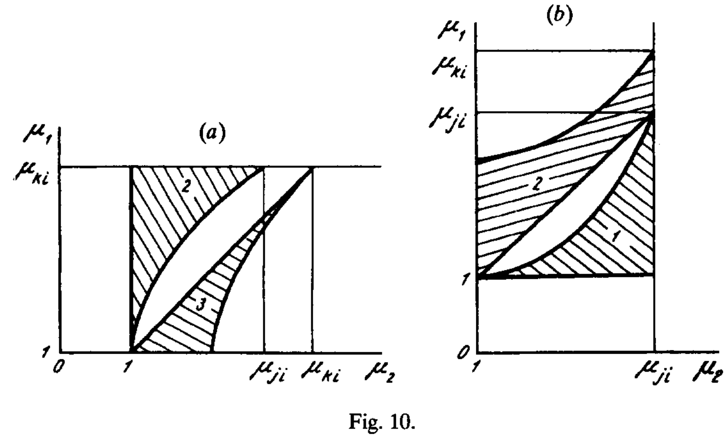

Fig. 10.

$$r_{21}^* = a\left(\frac{35}{6u}\right)^{1/2}(1+o(1)), \quad r_{32}^* = a\left(\frac{35}{6v}\right)^{1/2}(1+o(1))$$

$$r_{31}^* = a\left(\frac{35}{6u+6v}\right)^{1/2}(1+o(1)), \quad \bar{r}_{32}^* = a\left(\frac{35}{8v}\right)^{1/2}(1+o(1)) \tag{9.1}$$

$$\bar{r}_{21}^* = a\left(\frac{35}{8u}\right)^{1/2}(1+o(1)), \quad \bar{r}_{31}^* = a\left(\frac{35}{8v+8u}\right)^{1/2}(1+o(1))$$

Consequently, in this case one has $a/r \ll 1$ in the neighbourhood of the bifurcation points, and the phenomena listed above are maintained even if the dimensions of the body are much less than the distance between its mass centre and the attracting centre. This corroborates our investigation of the restricted formulation of the problem, in which the first two phenomena remain valid for the relative equilibria of a body whose inertia ellipsoid is nearly a sphere (see (6.5)).

Note that when $a/r \ll 1$ one usually uses the satellite approximation of the gravitational potential; under those circumstances, as we know [1], none of the above phenomena occurs. In particular, when the satellite approximation is used only trivial steady orientations exist (usually 24 of them).

When the exact expression for the gravitational potential is used and the unrestricted formulation of the problem is adopted, 72 steady orientations exist (at any rate, in our model problem): 24 trivial orientations (1.5), 24 "plane" ($\theta = 0$) non-trivial orientations (2.1), and 24 "three-dimensional" ($\theta \neq 0$) non-trivial orientations (2.2). In addition, if the inertia ellipsoid is almost a sphere (see (2.6)), then at least 32 additional steady motions "of general form" (3.1) exist (for arbitrary masses m_1, m_2 and m_3 that are sufficiently close together in value). These 104 steady orientations correspond to the 104 steady orientations described in [3] for the special case $m_1 = m_2 = m_3$; in fact, they yield the latter if one lets $u^2 + v^2 \to 0$. Finally, if $2m_2 > m_1 + m_3$, 32 more steady orientations "of general form" exist, which have no analogue in the special case $m_1 = m_2 = m_3$.

Similar conclusions hold for the restricted formulation of the problem. The only difference is that then all the relative equilibria are characterized by the condition $\theta \equiv 0$; as already pointed out, for the relative equilibria (5.3) and (6.1) corresponding to the "two-dimensional" ($\theta = 0$) steady motions (1.5) and (2.1), the reaction of the constraint $\theta = 0$ is zero, while for the relative equilibria (6.2) and (8.1), corresponding to the "three-dimensional" $\theta \neq 0$ steady motions (2.2) and (3.1), the reaction does not vanish (see the expressions for the undetermined multiplier λ).

In addition, when $r \gg a$ non-trivial steady motions and relative equilibria exist only for bodies whose ellipsoid of inertia is nearly an ellipsoid of revolution (in particular, a sphere).

We also note that steady motions of the body for which the plane of the orbit of the mass centre passes through the attracting centre exist (for $m_1 \neq m_2 \neq m_3$) only when one of the principal central axes of inertia is orthogonal to the orbital plane.

The research reported here was carried out with financial support from the Russian Foundation for Basic Research (93-013-16242) and the International Science Foundation (MAK 000).

REFERENCES

1. BELETSKII V. V., *Motion of an Artificial Satellite Relative to the Centre of Mass*. Nauka, Moscow, 1965.
2. PASCAL M., Sur le mouvement d'un triple batonnet dans un champ newtonien. *J. de Méc.* **11**, 1, 147–160, 1972.
3. SULIKASHVILI R. S., Steady motions of bodies admitting of symmetry groups of regular polyhedra in a Newtonian force field. *Prikl. Mat. Mekh.* **53**, 4, 582–586, 1989.
4. ABRAROVA Ye. V. and KARAPETYAN A. V., Steady motions of a rigid body in a central gravitational field. *Prikl. Mat. Mekh.* **58**, 5, 68–73, 1994.
5. ABRAROVA Ye. V., Stability of steady motions of a rigid body in a central field. *Prikl. Mat. Mekh.* **59**, 6, 84–91, 1995.

Translated by D.L.